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COSMOLOGICAL VARIATION OF G AND THE SOLAR LUMINOSITY

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ABSTRACT

We reexamine Teller's analysis of the effects of a varying gravitational constant G on the past solar luminosity. We show that if Newtonian gravitation is viewed as a nonrelativistic limit of Einstein's theory, there exists (1) a constraint between G and the total mass M of the Sun and (2) a change in the radiative energy density-temperature relation, which were not included in Teller's analysis and which change his result from $L \sim G^7$ (found to be unacceptable) to $L \sim$ constant, independently of how G might vary with time.

Subject headings: cosmology — gravitation — Sun: general

I. INTRODUCTION

The possibility that the gravitational constant G might have changed with cosmological time (Dirac 1937) has been discussed off and on for about 40 years.

Broadly speaking, the work and reasoning on this subject can be characterized by three attitudes: (1) the variation of G can be dismissed on the grounds that it is disallowed by Einstein equations, (2) it can be easily disproved at the Newtonian level using planetary data, or (3) it can be disproved with a calculation first performed by Teller (1948) on the past luminosity of the Sun. This last argument is actually the one most often quoted and referred to in the literature dealing with a varying G.

As for the first argument, we shall show in this paper that Einstein's equations do not actually require the constancy of G. For an object (like the Sun or a planet) of mass M, we shall show that the only constraint is $GM =$ constant and that no statement about G and M separately can be derived from the theory.

As an immediate application of the constancy of the product GM, it follows that planetary distances R and periods P, which, from Kepler's third law and the conservation of angular momentum per unit mass, can be shown to go like

$$R \sim \frac{1}{GM}, \qquad P \sim \frac{1}{(\mathrm{GM})^2}, \tag{1}$$

are *predicted* to be constant in time no matter what G does. Observational (radar) data can therefore be used to confirm (if $\dot{R}_{\rm obs} = 0$), or disprove (if $\dot{R}_{\rm obs} \neq 0$), equation (1), but not to put limits on $\dot{G}$ (Canuto, Hsieh, and Owen 1979*a*, *b*).

The third and final argument concerning the luminosity dependence on G is more delicate. We shall show that Teller's well-known result

$$L \sim \frac{(GM)^7}{M^2} \sim G^7 \tag{2}$$

should be replaced by

$$L \sim \frac{(GM)^7}{(GM)^2} \sim \text{constant}\,. \tag{3}$$

The appearance of the two extra powers of G in the denominator is due to a change (overlooked by Teller) in the relation between equilibrium radiative energy density and temperature. Finally, the last step in equation (3) is due to a dynamical constraint mentioned before, namely that the product GM must remain constant, independently of whether G varies or not. The results

$$R \sim \text{const.}, \quad P \sim \text{const.}, \quad L \sim \text{const.}, \tag{4}$$

are due to the fact that the way G enters in the Newtonian or Einsteinian formalism is such as to make the final expression for R, P, and L invariant under the transformation

$$G = \text{constant} \rightarrow G = G(t)\,. \tag{5}$$

The Newtonian framework is therefore unable to differentiate between the two possibilities, and, although it is legitimate to *choose* that G has remained constant, no argument can be adduced to demonstrate that G has not changed, nor can any observational data be used to put limits on the time variation of G.

However interesting the question of the time variation of G might be, we see we cannot treat it as long as we limit ourselves to the Newtonian and Einsteinian framework, where G is in fact a ghost that can be renormalized away entirely, as our analysis will show.

II. THE CONSTRAINT

Since the development of Lagrangian mechanics, it is well known that with dynamical laws are associated

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conservation laws. Thus, if the dynamics is changed, one must make sure, for consistency requirements, that the conservation laws are appropriately modified. For gravitation, this is most easily seen from Einstein's field equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} . \tag{6}$$

In keeping with Teller's spirit of modifying gravitational dynamics, we assume G to be a scalar function so that, in the Newtonian limit, the usual form of "hydrostatic equilibrium" equation with a slowly varying G is recovered.

Because of the Bianchi identity, the left-hand side of equation (6) has zero divergence; it follows that the right-hand side must satisfy the equation

$$(GT^{\mu\nu})_{;\nu} = 0 . \tag{7}$$

To understand the significance of this conservation law, we consider an ideal fluid with total energy density ρ, pressure p, and fluid velocity v, the energy momentum tensor of which can be written as

$$T_{\mu\nu} = (p + \rho)v_\mu v_\nu - pg_{\mu\nu} . \tag{8}$$

For the purpose of subsequent analysis, we shall separate ρ into ρ_0, the mass energy density, and u, the internal energy density. We also make use of the equation of state relating u and p so that

$$\rho = \rho_0 + u = \rho_0 + \frac{1}{\gamma - 1} p . \tag{9}$$

Contracting (7) with v_μ and using (8), we derive (for any quantity A, $\dot{A} \equiv A_{;\mu}v^\mu$)

$$\dot{\rho} + (p + \rho)v^\mu{}_{;\mu} = -\rho\dot{G}/G , \tag{10}$$

where we have used the equation of geodesic motion

$$v^\mu{}_{;\nu}v^\nu = 0 .$$

If we denote by V the comoving volume occupied by a fluid element, we have (Misner, Thorne, and Wheeler 1973, exercise [22.1])

$$v^\mu{}_{;\mu} = \dot{V}/V ,$$

so that (10) can be rewritten as

$$\dot{\rho} + (\rho + p)\dot{V}/V = -\rho\dot{G}/G . \tag{11}$$

When G is constant, equation (11) reduces to the well-known energy conservation equation (Weinberg 1972, eq. [14.2.19]). With the aid of (9), equation (11) can be further transformed to read

$$\rho_0 \frac{(G\rho_0 V)^\cdot}{(G\rho_0 V)} + u\frac{(uV^\gamma G)^\cdot}{(uV^\gamma G)} = 0 . \tag{12}$$

At this point, we recall that in addition to the equations of energy and momentum balance, another equation, expressing the conservation of rest mass or particle number, is usually postulated. In any (relativistic) theory with mass-energy equivalence, this equation must be compatible with the energy conservation equation, equation (12), in the sense that the two must be identical when a pressureless fluid is considered. Thus, from equation (12), with $u = 0$, we find that in a G-varying scheme the relation expressing the rest-mass or baryon number conservation must read

$$(G\rho_0 V)^\cdot = 0 .$$

Applying the above relation to a star, such as the Sun, we find

$$GM = \text{constant} , \tag{13}$$

where M is the total rest mass. This is one of the constraints referred to earlier.

To complete the analysis, we need the relation between the radiation energy density ρ_γ and the temperature T, which Teller took to be the standard one, $\rho_\gamma \sim T^4$. This relation, however, is not valid in a G-varying scheme, as can be seen from equation (12). For radiation (eq. [9]: $\rho \equiv \rho_\gamma$, $\rho_0 = 0$, and $\gamma = 4/3$), we obtain

$$\rho_\gamma \sim \frac{1}{G}\frac{1}{V^{4/3}} \sim \frac{1}{G}T^4 . \tag{14}$$

Using (13) and (14), we can now analyze the G-dependence of the solar luminosity L, a reasonable estimate of which is given by the ratio of the total radiation energy to the photon diffusion time τ, i.e. (R is the radius of the Sun),

$$L \sim \frac{R^3\rho_\gamma}{\tau} \sim \frac{1}{G}\frac{R^3T^4}{\tau} , \tag{15}$$

where

$$\tau = \frac{R}{c}\frac{R}{l} = \frac{R^2}{c}k\rho_0 , \tag{16}$$

l being the photon mean free path and k the opacity ($k = 1/l\rho_0$). Equations (16) and (15) yield ($M \sim \rho_0 R^3$)

$$L \sim \frac{1}{GM}\frac{(TR)^4}{k} \sim \frac{1}{GM}\frac{(GM)^4}{k} , \tag{17}$$

where we have used the hydrostatic equilibrium equation in the form ($\rho_0 \equiv mn$, n being the number density)

$$\frac{p}{R} = \frac{nkT}{R} \sim \frac{GM\rho_0}{R^2} . \tag{18}$$

Note that the presence of the factor G in (14) makes L depend on M only through the combination GM. In the case of electron scattering opacity, k = const., and so

$$L \sim (GM)^3 \sim \text{constant} \tag{19}$$

because of (13).

For the opacity considered by Teller (the modified Kramers opacity) we must be careful because (14) is involved in the derivation of k. Radiative equilibrium demands

$$j = k\rho_\gamma , \tag{20}$$

where j is the emission rate from the physical process under consideration (e.g., free-free transition). Because of (14), it follows that k is proportional to G. Teller's opacity law

$$k \sim \rho_0 T^{-3} \tag{21}$$

must therefore be changed to

$$k \sim G\rho_0 T^{-3}\,, \tag{22}$$

so that equation (17) becomes

$$L \sim \frac{1}{GM}\frac{(GM)^4}{G\rho_0}T^3 \sim \frac{1}{GM}\frac{(GM)^4}{GM}(TR)^3 \sim \frac{(GM)^7}{(GM)^2} \tag{23}$$

instead of Teller's equation (4)

$$L \sim (GM)^7/M^2\,. \tag{24}$$

The two factors of G in the denominator of (23) are both due to relation (14). On the basis of (13), we see that equation (23) yields

$$L \sim \text{const.}$$

We conclude therefore that within the Newtonian framework, the Sun's luminosity is constant in time, independently of whether G varies or not (provided that the modified Kramers opacity is used).

Contrary to Teller's analysis, *in which G and M appear in the final result with different powers*, in the present treatment they appear only through the combination GM, which is dynamically constrained to be constant. Such a combination deprives G of the identity needed to qualify as an independent variable; in fact, a redefinition of M can be made so as to make G disappear entirely from the problem, which is in effect what our analysis has shown. For this reason, it is often correctly stated that, within the Newtonian scheme, G does not actually exist and that talking about a variation of G is meaningless (McCrea 1979).

In the same spirit, attempts at using planetary physics (within the Newtonian scheme) cannot be expected to yield any information about a varying G since G, entering the problem *only* via Kepler's law, $\omega^2 R^3 = GM$, appears multiplied by M and because of (13) it drops out of the problem entirely (Canuto, Hsieh, and Owen 1979*a*, *b*).

These examples clearly indicate the existence of a dilemma: one is interested in the theoretical implications of a time-varying gravitational "constant," but at the same time one is prevented from obtaining the desired answer within the existent formalism: the Newtonian and Einsteinian schemes were in fact not constructed with such a possibility in mind.

This important question is bound to remain unanswered unless a new formalism is constructed which, in addition to all the successful features of the Einstein's theory, possess the extra latitude necessary to study the consequences of a possible varying G (Canuto, Hsieh, and Owen 1979*a*, *b*).

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